Baxter Q-operators for integrable DST chain.

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Abstract

Following the procedure, described in the paper [10], for the integrable DST chain we construct Baxter Q-operators [1] as the traces of monodromy of some M-operators, that act in quantum and auxiliary spaces. Within this procedure we obtain two basic M-operators and derive some functional relations between them such as intertwining relations and wronskian-type relations between two basic Q-operators.

1 Introduction

Integrable periodic quantum DST (Discrete Self-Trapping) chain is a quantum system described by the following hamiltonian (it corresponds to a certain set of parameters $\omega_0, \gamma, \epsilon$ and m_{ij} in the hamiltonian considered in [2])

$$H = \sum_{k=1}^{N} [\varphi_k^+ \varphi_k + (\varphi_k^+ \varphi_k)^2/2 + \varphi_{k+1}^+ \varphi_k], \tag{1}$$

where the canonical variables φ_k^+ and φ_k satisfy commutation relations $[\varphi_i, \varphi_j^+] = \delta_{ij}$ and periodic boundary condintions $\varphi_{k+N} = \varphi_k$, $\varphi_{k+N}^+ = \varphi_k^+$. The system can be considered within the framework of the quantum inverse scattering method or R-matrix method. There exists the Lax operator connected with DST chain. It acts in the tensor product of n-th quantum space and two-dimensional auxilliary space $\mathbb{C}^2$ (see [3], [5]):

$$L_n(x) = \begin{pmatrix} x - i/2 - i\varphi_n^+ \varphi_n & \varphi_n^+ \\ \varphi_n & i \end{pmatrix}, \tag{2}$$

where x is the spectral parameter. The fundumental relations for the matrix elements of the Lax operator could be written in the following R-matrix form:

$$R_{12}(x-y) L_n^1(x) L_n^2(y) = L_n^2(y) L_n^1(x) R_{12}(x-y), \tag{3}$$

where the indicies 1, 2 indicate different auxiliary spaces and $R-$matrix is given by

$$R_{12}(x) = x + iP_{12}, \tag{4}$$

where P_{12} is the permutation operator in the auxiliary spaces 1 and 2. The same intertwining relations are true for the monodromy matrix $T(x) = \prod_{n=1}^{N} L_n(x)$ (here the multipliers

are ordered from right to left), this in turn means that $[t(x),t(y)]=0$, where we denote $t(x)=TrT(x)$ (the trace is taken over the auxiliary space). Thus coefficients of the polynomials (over the spectral parameter) $t(x)$ form the family of commuting operators and the hamiltonian H and the number of particle operator belong to this family (namely, if we expand $t(x)=\sum\limits_{k=0}^{N}(x-i/2)^k H^{(k)}$, the number of particle operator $\hat{n}=\sum\limits_{k=1}^{N}\varphi_k^+\varphi_k=iH^{(N-1)}$ and hamiltonian $H=iH^{(N-1)}-(H^{(N-1)})^2/2+H^{(N-2)})$.

The eigenvectors and the eigenvalues of $t(x)$ could be constructed in the framework of ABA [8]. In this approach one considers the monodromy matrix given by

$$T(x)=\prod_{n=1}^{N}L_n(x)=\left(\begin{array}{cc}\hat{A}(x) & \hat{B}(x)\\ \hat{C}(x) & \hat{D}(x)\end{array}\right).$$

There exists the so-called Bethe vacuum: $C(x)\Omega=0(\Omega=\prod\otimes\omega_k$, where $\varphi_k\omega_k=0)$. Vectors $\hat{B}(x_1)...\hat{B}(x_l)\Omega$ will be eigenvectors of $t(x)=TrT(x)=\hat{A}(x)+\hat{D}(x)$ with eigenvalues

$$t(x)=(x-i/2)^N\prod_{j=1}^{l}\frac{(x-x_j+i)}{(x-x_j)}+i^N\prod_{j=1}^{l}\frac{(x-x_j-i)}{(x-x_j)}$$

provided that x_i obey the Bethe equations

$$\prod_{j=1}^{l}\frac{(x_i-x_j-i)}{(x_i-x_j+i)}=-\frac{(x_i-i/2)^N}{i^N}$$

So the polynomial $q(x)=\prod_{j=1}^{l}(x-x_j)$ satisfies Baxter equation:

$$t(x)q(x)=(x-i/2)^N q(x-i)+i^N q(x+i). \tag{5}$$

According to Baxter [1] let us define the operator $Q(x)$ such that:

$$t(x)Q(x)=(x-i/2)^N Q(x-i)+i^N Q(x+i), \tag{6}$$

and $[t(x),Q(y)]=0,\ [Q(x),Q(y)]=0$.

The model under consideration occupies an intermediate place between two other integrable models: the XXX spin chain and the Toda chain (Lax operators of these modeles are intertwined by the rational R-matrix (4) too). As in the case of XXX spin chain there is the Q-operator with polynomial eigenvalues in the spectral parameter. It corresponds to the ABA. If we consider the equation (5) as discrete analog of a second order differential equation and immediately there arises the question about the second solution of (5). These second solutions have been intensively discussed in [4, 9]. The eigenvalues of the second Q-operator for DST model are meromorphic functions (in the case of Toda chain there is no ABA but there exist two Q-operators: one with entire eigenvalues, second with meromorphic eigenvalues [10]). In the next part these two solutions of (6) will be constructed.

The existence of the second Q-operator, which is linear independent from the first one could be seen from the following simple consideration (similar disscussions for the case of XXX-spin chain see in [9]). Let us consider Baxter equation for the eigenvalue of the first

Q-operator - $q(x)$, which is a polinomial of degree n (in the case of DST chain n equals the eigenvalue of number of particle operator $\sum\limits_{i=1}^{N}\varphi_i^+\varphi_i$), and the eigenvalue of the trace of monodromy matrix $t(x)$, which is a polinomial of degree N:

$$t(x)q(x) = (x-i/2)^N q(x-i) + i^N q(x+i) \tag{7}$$

or

$$\frac{t(x)}{q(x+i)q(x-i)} = \frac{(x-i/2)^N}{q(x)q(x+i)} + \frac{i^N}{q(x)q(x-i)} \tag{8}$$

Multiplying this equation by $\Gamma^N(-i(x-i/2))$ we get:

$$\frac{t(x)\Gamma^N(-i(x-i/2))}{q(x+i)q(x-i)} = \frac{i^N\Gamma^N(-i(x+i/2))}{q(x)q(x+i)} + \frac{i^N\Gamma^N(-i(x-i/2))}{q(x)q(x-i)} \tag{9}$$

Let us denote

$$S(x) = \frac{i^N\Gamma^N(-i(x+i/2))}{q(x)q(x+i)}, \tag{10}$$

then

$$\frac{t(x)\Gamma^N(-i(x-i/2))}{q(x+i)q(x-i)} = S(x) - S(x-i). \tag{11}$$

$S(x)$ can be rewritten as follows:

$$S(x) = i^N\Gamma^N(-i(x+i/2))\Big[\frac{q_1(x)}{q(x+i)} + \frac{q_2(x)}{q(x)}\Big], \tag{12}$$

where $q_1(x)$ and $q_2(x)$ are polynomials of degree less than n. Substituting this expansion into Baxter equation (8) we get:

$$\frac{t(x)}{q(x+i)q(x-i)} = (x-i/2)^N\Big[\frac{q_1(x)}{q(x+i)} + \frac{q_2(x)}{q(x)}\Big] + i^N\Big[\frac{q_1(x-i)}{q(x)} + \frac{q_2(x-i)}{q(x-i)}\Big] \tag{13}$$

Since $t(x)$ is a polynomial we see that $(x-i/2)^N q_2(x) + i^N q_1(x-i) = r(x)q(x)$, where $r(x)$ is a polynomial with the degree less than N. Expressing then $q_1(x)$ via $q_2(x)$ and $r(x)$, let us substitute it in the expression for $S(x)$:

$$S(x) = i^N\Gamma^N(-i(x+i/2))r(x+i) + i^N\Gamma^N(-i(x+i/2))\frac{q_2(x)}{q(x)} - \Gamma^N(-i(x+3i/2))\frac{q_2(x+i)}{q(x+i)} \tag{14}$$

Now our task is to present $S(x)$ in the following form

$$S(x) = \frac{p(x+i)}{q(x+i)} - \frac{p(x)}{q(x)}, \tag{15}$$

and $p(x)$ will be the eigenvalue of the second Q-operator. Indeed from (10) we get

$$i^N\Gamma^N(-i(x+i/2)) = p(x+i)q(x) - p(x)q(x+i), \tag{16}$$

and $i^N\Gamma^N(-i(x+i/2))^N\Gamma^N(-i(x-i/2)) = p(x)q(x-i) - p(x-i)q(x)$. Multiplying the last equation by $(-i(x-i/2))^N$ and subtracting it from the previous one we see, that $p(x)$ satysfies the same Baxter equation:

$$t(x)p(x) = (x-i/2)^N p(x-i) + i^N p(x+i). \tag{17}$$

Thus the next step is to find the function $g(x)$ such that

$$g(x+i) - g(x) = i^N\Gamma^N(-i(x+i/2))r(x+i). \tag{18}$$

Let us look for $g(x)$ in the following form:

$$g(x) = \sum_{k=0}^{\infty} f(-ix-k). \tag{19}$$

In this case $g(x+i) - g(x) = -f(-ix)$, and we see that if

$$f(-ix) = -i^N\Gamma^N(-i(x+i/2))r(x+i),$$

then

$$g(x) = -i^N \sum_{k=0}^{\infty} \Gamma^N(-i(x+i/2)-k)r(x+i-ik), \tag{20}$$

and the desired eigenvalue will be given

$$p(x) = g(x)q(x) - i^N\Gamma^N(-i(x+i/2))q_2(x) \tag{21}$$

Apparently (21) is a meromorphic function with respect to the spectral parameter x which has the poles at the integer values of $y = -ix + 1/2$ (the convergency of the series for $g(x)$ at $-ix + 1/2 \neq \mathbb{Z}$ is provided by the term $-k$ in the gamma function argument).

As an illustration consider a simple example for the concrete polynomial solution of Baxter equation for the case of two degrees of freedom

$$q(x) = x^2 - 2ix + 1/4.$$

This solution corresponds to the Bethe vector $\frac{1}{\sqrt{2}}(|2,0> - |0,2>)$ and the eigenvalue of

$$t(x) = x^2 - 3ix - 9/4.$$

Here we have introduced following notation for the vectors of quantum space:

$$|k_1, k_2 >= (\varphi_1^+)^{k_1}(\varphi_2^+)^{k_2}|0,0>,$$

where $|0,0>$ is the Bethe vacuum ($\varphi_1|0,0>= \varphi_2|0,0>= 0$).
The explicit construction of the polynomials $q_1,\ q_2,\ r$ using the method described above gives:

$$q_1(x) = -i/2x + 1/4, \quad q_2(x) = i/2x + 3/4, \quad r(x) = i/2x + 1/4.$$

And for the eigenvalue of the second Q-operator we obtain

$$p(x) = (x^2 - 2ix + 1/4)\sum_{k=0}^{\infty} \Gamma^2(-ix + 1/2 - k)(i/2x + k/2 - 1/4) +$$
$$+\Gamma^2(-ix + 1/2)(i/2\ x + 3/4)$$

2 Basic Q-operators for the DST model

In the present paper we shall construct the basic Q-operators using the method described in [10]. In this approach the $Q^{(1,2)}$-operators are the traces of monodromies $\hat{Q}^{(1,2)}$ of appropriate $M_n^{(1,2)}$-operators acting in n-th quantum space and in the auxilliary space Γ, which is the representation space of Heisenberg algebra $[\rho,\rho^+]=1$. As we shall need to consider the product of $L(x)M^{(1,2)}(x)$, the mutual auxilliary space for this object are $\Gamma\otimes\mathbb{C}^2$, where we can introduce the projectors:

$$\Pi^+_{ij}=\left(\begin{array}{c}1\\ \rho\end{array}\right)_i\frac{1}{(\rho^+\rho+1)}(1,\rho^+)_j,\quad \Pi^-_{ij}=\left(\begin{array}{c}-\rho^+\\ 1\end{array}\right)_i\frac{1}{(\rho^+\rho+2)}(-\rho,1)_j \tag{22}$$

According to the method of [10] we impose the condition that the products $L(x)M(x)$ and $M(x)L(x)$ have triangle forms in the sense of projectors $\Pi^\pm$:

$$\left\{\begin{array}{l}\Pi^-_{ik}\left(L_n(x)\right)_{kl}M_n^{(1)}(x)\Pi^+_{lj}=0\\ \Pi^+_{ik}M_n^{(1)}(x)\left(L_n(x)\right)_{kl}\Pi^-_{lj}=0\end{array}\right., \tag{23}$$

for $M_n^{(1)}(x)$ and

$$\left\{\begin{array}{l}\Pi^+_{ik}\left(L_n(x)\right)_{kl}M_n^{(2)}(x)\Pi^-_{lj}=0\\ \Pi^-_{ik}M_n^{(2)}(x)\left(L_n(x)\right)_{kl}\Pi^+_{lj}=0,\end{array}\right. \tag{24}$$

for $M_n^{(2)}(x)$. Consider first the system for $M^{(1)}$. It follows from the first equation in (23) that:

$$\left\{\begin{array}{c}M^{(1)}(x)\left(\begin{array}{c}1\\ \rho\end{array}\right)_i=\tilde{L}(x)_{ij}\left(\begin{array}{c}1\\ \rho\end{array}\right)_j A^{(1)}(x)\\ B^{(1)}(x)\left(\begin{array}{c}1\\ \rho\end{array}\right)_i=\tilde{L}(x+i)_{ij}\left(\begin{array}{c}1\\ \rho\end{array}\right)_j M^{(1)}(x)\end{array}\right., \tag{25}$$

where we introduce

$$\tilde{L}(x)=\left(\begin{array}{cc}i & -\varphi^+\\ -\varphi & x-3i/2-\varphi^+\varphi\end{array}\right),$$

with properties $L(x)\tilde{L}(x)=i(x-i/2)\cdot I$ (I is the identity matrix) and $L(x)+\tilde{L}(x+i)=TrL(x)\cdot I$ (this identity provides the argument shift in (25) and leads to the finite differences equation). System (25) has the solution of the form $B^{(1)}(x)=cM^{(1)}(x+i)$, $A^{(1)}(x)=c^{-1}M^{(1)}(x)$, where c is a number. Let us choose $c=i$. Along with the analogous consideration of the triangularity condition for right multiplication $\Pi^+_{ik}M_n^{(1)}(x)\left(L_n(x)\right)_{kl}\Pi^-_{lj}=0$ it leads to the system:

$$\left\{\begin{array}{c}\tilde{L}(x+i)_{ij}\left(\begin{array}{c}1\\ \rho\end{array}\right)_j M^{(1)}(x)=M^{(1)}(x+i)\left(\begin{array}{c}1\\ \rho\end{array}\right)_i\\ M^{(1)}(x)L(x)_{ij}\left(\begin{array}{c}-\rho^+\\ 1\end{array}\right)_j=i\left(\begin{array}{c}-\rho^+\\ 1\end{array}\right)_i M^{(1)}(x+i)\end{array}\right., \tag{26}$$

For $M^{(2)}$ we get:

$$\left\{\begin{array}{c}\tilde{L}(x+i)_{ij}\left(\begin{array}{c}-\rho^+\\ 1\end{array}\right)_j M^{(2)}(x)=M^{(2)}(x+i)\left(\begin{array}{c}-\rho^+\\ 1\end{array}\right)_i\\ M^{(2)}(x)L(x)_{ij}\left(\begin{array}{c}1\\ \rho\end{array}\right)_j=i\left(\begin{array}{c}1\\ \rho\end{array}\right)_i M^{(2)}(x+i)\end{array}\right.. \tag{27}$$

The full multiplication rules have the following form:

$$
\begin{aligned}
&(L_n(x))_{ij}\, M_n^{(1)}(x) = \begin{pmatrix} 1 \\ \rho \end{pmatrix}_i M_n^{(1)}(x-i)\frac{1}{\rho^+\rho+1}(1,\rho^+)_j + \\
+ &\begin{pmatrix} -\rho^+ \\ 1 \end{pmatrix}_i \frac{1}{\rho^+\rho+2} M_n^{(1)}(x+i)(-\rho,1)_j + \Pi^+_{ik}\,(L_n(x))_{kl}\, M_n^{(1)}(x)\Pi^-_{lj}
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
&(L_n(x))_{ij}\, M_n^{(2)}(x) = \begin{pmatrix} 1 \\ \rho \end{pmatrix}_i \frac{1}{\rho^+\rho+1} M_n^{(2)}(x+i)(1,\rho^+)_j + \\
+ &\begin{pmatrix} -\rho^+ \\ 1 \end{pmatrix}_i M_n^{(2)}(x-i)\frac{1}{\rho^+\rho+2}(-\rho,1)_j + \Pi^-_{ik}\,(L_n(x))_{kl}\, M_n^{(2)}(x)\Pi^+_{lj}
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
&M_n^{(1)}(x)\,(L_n(x))_{ij} = \begin{pmatrix} 1 \\ \rho \end{pmatrix}_i \frac{1}{\rho^+\rho+1} M_n^{(1)}(x-i)(1,\rho^+)_j + \\
+ &\begin{pmatrix} -\rho^+ \\ 1 \end{pmatrix}_i M_n^{(1)}(x+i)\frac{1}{\rho^+\rho+2}(-\rho,1)_j + \Pi^-_{ik}\,(L_n(x))_{kl}\, M_n^{(1)}(x)\Pi^+_{lj}
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
&M_n^{(2)}(x)\,(L_n(x))_{ij} = \begin{pmatrix} 1 \\ \rho \end{pmatrix}_i M_n^{(2)}(x+i)\frac{1}{\rho^+\rho+1}(1,\rho^+)_j + \\
+ &\begin{pmatrix} -\rho^+ \\ 1 \end{pmatrix}_i \frac{1}{\rho^+\rho+2} M_n^{(2)}(x-i)(-\rho,1)_j + \Pi^+_{ik}\,(L_n(x))_{kl}\, M_n^{(2)}(x)\Pi^-_{lj}
\end{aligned}
\tag{31}
$$

We do not consider the irrelevant structures of the last terms in the rhs of these rules. Apparently, the triangle structure (23,24) will be valid also for products of L_n and M_n, as the quantum operators with different n commute with each other, therefore these relations guarantee that both the traces of monodromies (if they exist)

$$
Q^{(1,2)}(x) = Tr\hat{Q}^{(1,2)}(x) = Tr\prod_{k=1}^{N} M_k^{(1,2)}(x),
\tag{32}
$$

satisfy Baxter equation (6).

To solve the equations (26,27) we shall use the holomorphic representation for the operators ρ, ρ^+. Let the operator ρ^+ be the operator of multiplication: $(\rho^+\psi)(\alpha) = \alpha\psi(\alpha)$, while ρ is the operator of differentiation $(\rho\psi)(\alpha) = \frac{\partial}{\partial\alpha}\psi(\alpha)$. The action of an operator in the holomorphic representation is defined by its kernel:

$$
(\hat{M}\psi)(\alpha) = \int d^2\mu(\beta) M(\alpha,\bar{\beta})\psi(\beta),
\tag{33}
$$

where the measure is defined as follows: $d^2\mu(\beta) = e^{-\beta\bar{\beta}}d\beta d\bar{\beta}$.

In this representation the operators, which satisfy the systems (26,27) have the following forms:

$$
\begin{aligned}
M^{(1)}(x,\alpha,\bar{\beta}) &= e^{-i\bar{\beta}\varphi^+}\frac{\Gamma(-i(x-i/2))}{\Gamma(-\varphi^+\varphi - i(x-i/2))}e^{-i\alpha\varphi} \\
M^{(2)}(x,\alpha,\bar{\beta}) &= e^{i\alpha\varphi}e^{i\pi\varphi^+\varphi}\Gamma(-\varphi^+\varphi - i(x-i/2))e^{i\bar{\beta}\varphi^+}
\end{aligned}
\tag{34}
$$

In order to find the monodromy $\hat{Q}(x,\alpha,\bar{\beta})^{(1,2)}$ one has to perform an ordered multiplication of $M^{(1,2)}$- operators:

$$\hat{Q}^{(i)}(x,\alpha,\bar{\beta}) = \int \prod_{i=1}^{N-1} d^2\mu(\gamma_i) M_N^{(i)}(x,\alpha,\bar{\gamma}_{N-1}) M_{N-1}^{(i)}(x,\gamma_{N-1},\bar{\gamma}_{N-2}) \tag{35}$$

$$\cdots \times M_2^{(i)}(x,\gamma_2,\bar{\gamma}_1) M_1^{(i)}(x,\gamma_1,\bar{\beta}). \tag{36}$$

Taking the trace of $\hat{Q}^{(1,2)}$ over the auxiliary space we obtain $Q^{(1,2)}-$operators. The trace of an operator Q in the holomorphic representation is given by

$$TrQ = \int \mathrm{d}^2\mu(\alpha)\hat{Q}(\alpha,\bar{\alpha}), \tag{37}$$

where $\hat{Q}(\alpha,\hat{\alpha})$ is the kernel of $\hat{Q}$.

The eigevalues $Q^{(1)}(x)$ are polinomials in the spectral parameter x. It can be seen from the action of $Q^{(1)}$ onto the basic vectors $\mid n_1, n_2, ..., n_N >= (\varphi_1^+)^{n_1}(\varphi_2^+)^{n_2}...(\varphi_N^+)^{n_N}|0>$, where $|0>$ is the Bethe vacuum: $\varphi_k|0>=0,\ k=1..N$:

$$Q^{(1)}(x)|n_1,...,n_N> = \sum_{m_1,...,m_N=0}^{n_1,...,n_N} \prod_{k=1}^{N} \frac{(-1)^{m_k}}{m_k!}$$
$$\frac{\Gamma(-i(x-i/2))}{\Gamma(-i(x-i/2)-n_k+m_k)} \frac{n_k!}{(n_k-m_k)!}|...,n_k-m_k+m_{k-1},...> \tag{38}$$

We see $Q^{(1)}(x)$ leaves the subspace of vectors with a common particle number $n = n_1+n_2+ ...+n_N$ invariant, and all matrix elements of $Q^{(1)}$ are polinomials in x. We shall see below that $[Q^{(1)}(x_1), Q^{(1)}(x_2)] = 0$, so the eigenvalues of $Q^{(1)}$ are polinomials in x too. Constructed in [5] Q-operator corresponds to $Q^{(1)}$. Its action onto the basic vectors (in the paper [5] the coordinate representation has been choosen for the quantum operators with the basic vectors: $x_1^{n_1}\dots x_n^{n_N}$) is similar to $Q^{(1)}$ in (38).

For comparison we give also the action of $Q^{(2)}$ onto the same basic vectors:

$$Q^{(2)}(x)|n_1,...,n_N> =$$
$$= e^{i\pi n}\cdot \sum_{m_1,...,m_N=0}^{\infty} \prod_{k=1}^{N} \Gamma(-ix-1/2-n_k-m_{k-1})\cdot$$
$$\cdot\frac{(m_{k-1}+n_k)!}{m_k!(n_k+m_{k-1}-m_k)!}|...,n_k+m_{k-1}-m_k,...> . \tag{39}$$

Here the summation in contrast to $Q^{(1)}$ is taken over an infinite set of m_k, restricted however by the condintions $m_k - m_{k-1} \le n_k$.

Let us notice that in some realizations of quantum and axilliary operators there may appears the factorization of Q-operators first considered by Bazhanov and Stroganov [7] (see also [6]), if, for example, we choose the coordinate representaion for quantum and auxiliary operators, we will get the following factorized form for Q-operator [5, 6]:

$$Q(x_1,...,x_N,x'_1,...,x'_N) = \prod_{k=1}^{k=N} q_k(x_k,x'_{k+1},x'_k) \tag{40}$$

In the paper [5] one of the Q-operators in the form (40) was constructed. It is also possible to construct the second Q-operator (it was also notice in [5]) in the same factorized form without use of an axilliary space. However from the point of view of approach [10] the origin of such kind of factorization is not clear. and trace of factorized defined by in complex canonical onto also above $[Q(x), t(y)] = 0$, these

In the simplest case of one quantum degree of freedom $N = 1$ we obtain $(n = \varphi^+\varphi)$

$$Q^{(1)}(x) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \frac{\Gamma(-ix+1/2)}{\Gamma(-ix+1/2-n-k)} \tag{41}$$

$$Q^{(2)}(x) = e^{i\pi n} \sum_{m=0}^{\infty} \frac{(n+m)!}{m!n!} \Gamma(-ix-1/2-n-m). \tag{42}$$

As we have expected the eigenvalues of $Q^{(1)}$ are polynomial of degree n and the eigenvalues of $Q^{(2)}$ are meromorphic functions over the spectral parameter.

It is possible to find the solutions of (26,27) in the form:

$$\begin{aligned} M^{(1)}(x,\rho) &= \mathrm{P}^{\rho\varphi}(i-\varphi\rho^+)^{-i(x-i/2)} e^{-x\pi/2} \\ M^{(2)}(x,\rho) &= \Pi^{\rho\varphi}\Gamma(-\rho^+\rho - \varphi^+\varphi - i(x-i/2)) \end{aligned} \tag{43}$$

Where $\mathrm{P}^{\rho\varphi}$ is an operator equal:

$$\mathrm{P}^{\rho\varphi} = exp\left[\pi/2(\varphi^+\rho - \varphi\rho^+)\right] exp\left[i\pi/2(\rho^+\rho - \varphi^+\varphi)\right].$$

It has properties of the permutation operator:

$$\begin{aligned} &\mathrm{P}^{\rho\varphi}\varphi = i\rho\mathrm{P}^{\rho\varphi}, \quad \mathrm{P}^{\rho\varphi}\varphi^+ = -i\rho^+\mathrm{P}^{\rho\varphi}, \\ &\varphi\mathrm{P}^{\rho\varphi} = -i\mathrm{P}^{\rho\varphi}\rho, \quad \varphi\mathrm{P}^{\rho\varphi^+} = i\mathrm{P}^{\rho\varphi}\rho^+, \end{aligned} \tag{44}$$

$\Pi^{\rho\varphi}$ is an operator with the following properties:

$$\varphi\Pi^{\rho\varphi} = -i\rho\Pi^{\rho\varphi}, \quad \Pi^{\rho\varphi}\varphi^+ = -i\Pi^{\rho\varphi}\rho^+,$$

however

$$\varphi^+\Pi^{\rho,\varphi} = i[\Pi^{\rho,\varphi}, \rho^+], \quad \Pi^{\rho,\varphi}\varphi = -i[\Pi^{\rho,\varphi}, \rho].$$

The explicit expression for $\Pi^{\rho,\varphi}$ is

$$\begin{aligned} \Pi^{\rho\varphi} = \quad & \left[1 + \sum_{k=1}(i\varphi\rho^+)^k \frac{\Gamma(\rho^+\rho+1)}{\Gamma(\rho^+\rho+k+1)} + \sum_{k=1}(i\varphi^+\rho)^k \frac{\Gamma(\varphi^+\varphi+1)}{\Gamma(\varphi^+\varphi+k+1)}\right] \cdot \\ & \cdot \frac{\Gamma(\rho^+\rho+\varphi^+\varphi+1)}{\Gamma(\rho^+\rho+1)\Gamma(\varphi^+\varphi+1)} e^{i\pi\varphi^+\varphi}. \end{aligned} \tag{45}$$

The operators $M_n^{(1,2)}$ and $L_n(x)$ satisfy certain intertwining relations which will imply the mutual commutativity of Q-operators and the transfer matrix. Here we will present the intertwining relations without its derivation because the method used and the intertwining R-matrix are the same as in the case of the Toda chain [10].

They are: 1) for the Lax and M-operators:

$$R^{(i)}_{kl}(x-y)\,(L_n(x))_{lm}\,M^{(i)}_n(y) = M^{(i)}_n(y)\,(L_n(x))_{kl}\,R^{(i)}_{lm}(x-y) \tag{46}$$

The corresponding R-matrices are:

$$R^{(1)}_{kl}(x-y) = \begin{pmatrix} x-y+i\rho^+\rho & -i\rho^+ \\ -i\rho & i \end{pmatrix} \tag{47}$$

$$R^{(2)}_{kl}(x-y) = \begin{pmatrix} i & i\rho^+ \\ i\rho & x-y+i+i\rho^+\rho \end{pmatrix} \tag{48}$$

Note that in the paper [5] the equation (46) with $R^{(1)}$ was considered as the defining equation for local M-operators and the trace of their monodromy is the Q-operator.

2) for $M^{(1)}$- and $M^{(2)}$-operators acting in different auxiliary spaces

$$M^{(1)}_n(x,\rho)M^{(2)}_n(y,\tau)R^{12}(x-y) = R^{12}(x-y)M^{(2)}_n(y,\tau)M^{(1)}_n(x,\rho), \tag{49}$$

where the intertwining matrix R^{12} is defined by the kernel in the holomorphic representation

$$R^{12}(x,\alpha,\bar\beta;\gamma,\bar\delta) = \sum_{n=0} \frac{\big((\alpha-\gamma)(\bar\beta-\bar\delta)\big)^n}{n!\Gamma(-ix+n+1)}, \tag{50}$$

here $\alpha,\bar\beta$ and $\gamma,\bar\delta$ are the holomorphic variables for the representation space for pairs of operators ρ,ρ^+ and τ,τ^+.

3) for $M^{(1)}$-operators acting in different auxiliary spaces:

$$R^{(11)}(x-y)M^{(1)}(x,\rho)M^{(1)}(y,\tau) = M^{(1)}(y,\tau)M^{(1)}(x,\rho)R^{(11)}(x-y), \tag{51}$$

where

$$R^{(11)}(x) = P_{\rho\tau}(1+\rho^+\tau)^{-ix}, \tag{52}$$

here $P_{\rho\tau}$ is the operator of permutation of the auxiliary spaces.

4) for $M^{(2)}$-operators acting in different auxiliary spaces:

$$R^{(22)}(x-y)M^{(2)}(x,\rho)M^{(2)}(y,\tau) = M^{(2)}(y,\tau)M^{(2)}(x,\rho)R^{(22)}(x-y), \tag{53}$$

where

$$R^{(22)}(x) = P_{\rho\tau}(1+\tau^+\rho)^{-ix}. \tag{54}$$

These intertwining relations lead to the mutual commutativity of Q-operators and t-matrix.

$$[t(x),Q^{(i)}(y)] = 0, \quad [Q^{(i)}(x),Q^{(j)}(y)] = 0, \quad i,j=1,2. \tag{55}$$

So far we have constructed two solutions of the operator Baxter equation. Now we are going to establish linear independence of these solutions defining the one-parametric family of a finite difference analogues of the Wronsky determinant

$$W_m = Q_1(x-im)Q_2(x+i) - Q_1(x+i)Q_2(x-im) \tag{56}$$

where m is a non-negative integer. Consider properties of these objects following directly from Baxter equation:

$$\begin{aligned} t(x)Q_1(x) &= (x-i/2)^N Q_1(x-i) + i^N Q_1(x+i) \\ t(x)Q_2(x) &= (x-i/2)^N Q_2(x-i) + i^N Q_2(x+i). \end{aligned} \tag{57}$$

Multiplying first equation by $Q_2(x)$, second by $Q_1(x)$, and substracting one from another we see that

$$(x-i/2)^N W_0(x-i) = -i^N W_0(x),$$

so W_0 necessarily has the factor $\Gamma^N(-i(x+i/2))$. Multiplying then the first equation by $Q_2(x-i)$ and the second by $Q_1(x-i)$ we get

$$W_1(x) = (-i)^N t(x) W_0(x-i).$$

In the general case of non-negative integer m multiplying the first equation by $Q_2(x-im)$ and the second equation by $Q_1(x-im)$, we obtain:

$$(x-i/2)^N W_{m-2}(x-2i) + i^N W_m(x) = t(x) W_{m-1}(x-i). \tag{58}$$

These identities completely define all W_m provided W_0 is known. These argumentations make sense in the presence of two solutions of Baxter equation with $W_0 \neq 0$ identically. The eigenvalues of the transfer matricies $t_l(x)$-traces of monodromy matrixes in auxiliary spaces of spin $l = m/2$ satisfies the recurrent relations similar to (58). The family of t_l could be obtained with the help of the expression for the Lax operator in the auxiliary space of spin l:

$$L_l(x) = i^{2l} e^{-il^+\varphi^+} \frac{\Gamma(l^3 - ix)}{\Gamma(-ix-l)} e^{-il^-\varphi}. \tag{59}$$

Here operators l^k $(k = \pm,\ 3)$ are the operators of spin l and the factor $i^{2l}\Gamma^{-1}(-i(x+i/2))$ is introduced in order that in the cases of $l=0$ and $l=1/2$ we will obtain correspondingly $L_0 = 1$ and $L_{1/2}(x) = L(x)$-the Lax operator (2). For the operators considered in the Introduction the relation (16) gives

$$W_0(x) = i^N \Gamma^N(-i(x+i/2)).$$

An explicit calculation of W_0 for the solutions constructed in Section 2 using the method, described in the paper [10] gives

$$W_0(x) = e^{i\pi\hat{n}} \Gamma^N(-i(x+i/2)), \tag{60}$$

where $\hat{n}$ is the number of particles operator. And for this pair it follows that:

$$W_m(x) = e^{i\pi\hat{n}} (-i)^{2lN} \Gamma^N(-ix-1/2) t_l(x) \tag{61}$$

Finally we arrive at the following general Wronskian-type relations:

$$Q_1(x-im)Q_2(x+i) - Q_1(x+i)Q_2(x-im) = e^{i\pi\hat{n}} (-i)^{2lN} \Gamma^N(-ix-1/2) t_l(x). \tag{62}$$

3 Conclusion

In the present paper we have constructed the basic Q-operators in the form of traces of monodromies of basic local M-operators for the case of DST integrable model. The obtained Q-operators are presented in the form of formal series over the canonical operators φ_k, φ_k^+ and have well defined action onto vectors of quantum space. The intertwining relations indicating the mutual commutativity of $t(x)$ and Q-operators are derived. Obtained are functional relations of wronskian-type showing the linear independence of Q-operators and connection between the Q-operators and the transfer matrices in the auxiliary spaces of higher spins.

Let us notice some unsolved problems. It would be interesting to find Q-operators for small numbers of freedom degrees as functions of the family of commuting operators, connected with $t(x)$. The origin of the factorizations a lá Pasquer-Gaudin, which appear in some representations [6, 5] is not clear.

The described in the present method makes it possible to find M-operators in the most interesting case of the XXX-spin chain (they coincide with the Lax operators $L(x)$ and $\tilde{L}(x)$ for DST model with interchaged quantum and auxiliary spaces), but the traces of their monodromy diverge. However, there exists the procedure of Q-operator construction, analogous to the described in [6] one, for XXX SL(2,$\mathbb{C}$) spin chain [11]. So the case of XXX-spin chain deserves further investigation.

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